**Abstract**

Existing theories describing faceted growth and ablation of ice crystals have significant limitations in terms of the connection to underlying physical processes. In filling that knowledge gap, the present model provides insights and predictive power about faceted growth and ablation that have not been possible previously. For example, the model exhibits Turing-like pattern behavior, in the sense that the horizontal distance between successive ice layers atop a growing facet is predicted to increase in proportion to the square root of the surface diffusivity.

1. **Prior theories of faceted ice crystal growth and ablation**

In the BCF model, water vapor molecules land on a crystalline surface, then move around on that surface until they either become part of the lattice, or else detach from the surface and re-enter the vapor phase. As intuitively appealing as that model may be, it cannot describe the surface of real ice in the temperature 240 K to melting, because in that temperature range ice surfaces are covered by a quasi-liquid layer (QLL).

The quasi-liquid continuum model introduced by some of the authors in 2016 (N2016) recasts the problem of an ice surface covered by a QLL by designating two mesoscale variables, and (see Fig. 1), which represent the total thickness of the ice surface and the thickness of the quasi-liquid part of the surface, respectively.

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| **Figure 1**. Visual representation of mesoscale variables , , and , and processes affecting them, in the N2016 (and present) model. Dashed arrows represent processes affecting how these variables evolve over time. |

The dynamics of these two variables, according to N2016, are governed by a pair of reaction-diffusion differential equations that represent the three processes indicated in Fig. 1, namely, (i) vapor deposition and ablation to and from the QLL, (ii) horizontal surface diffusion of the QLL across the facet, and (iii) interconversion of QLL molecules to/from the underlying ice.

The main insight afforded by N2016 is that it provides a mechanism by which faceted ice crystal growth occur within the framework of a QLL-covered ice surface. That mechanism, termed “diffusive slowdown,” is an emergent property the reaction-diffusion equations, in which the surface structures itself in a way that the excess vapor available at corners of a growing facet are converted into ice less efficiently, compared to facet centers.

N2016 suffered from several limitations, however, of which the most important for our present purpose is that the time scale of process (iii), the interconversion of quasi-liquid and ice, was fixed relative to processes (i) and (ii); in real crystal facets, these relative time scales may vary from facet to facet, or as a function of temperature. The revised model presented here corrects this deficiency.

1. **A revised quasi-liquid reaction-diffusion model**

The present theory has much in common with N2016, beginning with its representation of an ice surface defined by the two mesoscale variables and three processes shown in Fig. 1. The governing reaction-diffusion equation is

(1a)

(1b)

Some notes about this model are as follows, with differences between it and N2016 noted:

1. represents the idea that surface diffusion depends on the thickness of the quasi-liquid only; the underlying ice is considered immobile on time scales considered here.
2. is the rate at which vapor-phase water molecules collide with the quasi-liquid; it is assumed that these stick, on a picosecond time scale, with 100% efficiency.
3. defines the thickness of quasi-liquid when it is in equilibrium with the underlying ice. Here (as in N2016) we use the sinusoidal form

(2)

It is evident from this equation that this thickness varies continuously from a thin state (“surface I”) with thickness , and a thick state (“surface II”) with thickness . These are indicated schematically in Fig. 1.

1. The model assumes that surface II is more volatile than faster surface I. (Equivalently, the equilibrium vapor pressure of surface II is higher than the equilibrium vapor pressure of surface I). Therefore, the net surface supersaturation at a given point on the surface, designated as in Eq. 1a, is not simply a function of the overlying water vapor concentration, but also of the state of the QLL at any given point: a given concentration of water vapor will “look” more supersaturated over a part of the QLL characterized by surface I, and less supersaturated to those parts characterized by surface I.

To compute , we define a variable that quantifies the degree to which a given surface is similar to surface I or II,

(3)

With this definition, a point on the surface whose thickness equals that of surface I will have , while a point whose thickness equals that of surface II will have . We then express as

(4)

where is a measure of the difference in the equilibrium vapor pressure of surfaces I and II, and is the supersaturation relative to surface I. We will assume here that both (a scalar quantity) and are fixed parameters of a given trajectory. Forms of the latter are described in more detail below. Eq. 4 is at slight variance with, and simpler than, the corresponding expression in N2016.

1. is a first-order relaxation constant describing the time scale at which quasi-liquid/ice equilibrium is achieved. That is, if we imagine an initial situation having an amount of quasi-liquid given by , then equilibration after a time occurs according to

(5)

If one takes the time derivative of Eq. 5, and assumes that is small, the second term on the right-hand side of Eq. 1b results.

Equations 5 and 1b represent the primary departure of the present model from N2016. With this revision, we are able to specify the rate of quasi-liquid/ice equilibration relative to processes (i) and (ii). Specifying a small value for , for example, would represent the idea that quasi-liquid/ice equilibration is fast compared to diffusion and exchanges with the vapor phase, while large would mean the opposite. We do not have reliable independent guides for determining , but we do have a guidepost: because the “diffusive slowdown” mechanism for stabilization of faceted ice growth described in N2016 required that quasi-liquid/ice equilibration be slow compared to surface diffusion, we should not be surprised if we find that large leads to stable growth scenarios. We return to this topic below.

1. **Informing theory with SEM experiments**

Environmental SEM imaging of ice crystals has seen considerable activity in recent years. Those techniques offer “mesoscale” resolution of ice surfaces – i.e., as small as micrometers – with conditions that are adjustable in real time: one can use the technique to grow crystals, and ablate them, by manipulating the temperature and pressure inside the SEM chamber. In tandem with those developments are algorithms that permit quantitative reconstruction of the surface morphology (also at mesoscale resolution).

Together – in a process we will refer to as “SEM/GNBF retrieval” – we can examine not just static ice surface morphology, but dynamic effects, e.g., way the way ice surface morphologies evolve in response to user-controllable temperature and water vapor concentration. SEM/GNBF retrievals therefore provide unique opportunities for constraining and evaluating predictions of the quasi-liquid reaction-diffusion model. Questions we seek answers to include:

1. *Existence of faceted ablation*. Is there such a thing as faceted ablation, and if so, does the model support such a phenomenon?
2. *Facet concavity and convexity*. What does concavity of convexity of a facet tell us about the conditions surrounding a crystal?
3. *Dependence of surface morphology on supersaturation, diffusion coefficient, and other parameters.* Is it possible to make general statements about how surface morphology varies as a function of these parameters?
4. *Resilience*. SEM observations show that faceting is resilient in the sense that a crystal with rough surface morphology can be restored to smooth morphology when exposed to highly supersaturated conditions. Is the model similarly resilient?
5. *Does facet roughness have an intrinsic characteristic length scale*? More particularly, is there a difference between the roughness that appears under supersaturated conditions, vs subsaturated conditions, and if so, does the model help us understand that difference?
6. *What about differences between facets*?
7. **Implementation details**

Python, accelerated with Numby. Surface morphology extraction using the GNBF formalism …

Figure 2 displays examples of supersaturation profiles used to parameterize solutions to the model. Why do they have these shapes? Well, …

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| **Figure 2**. Supersaturation and subsaturation profiles. |

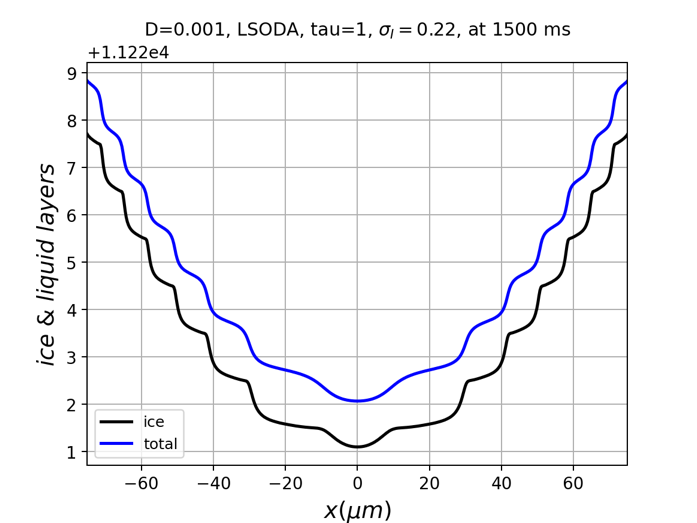
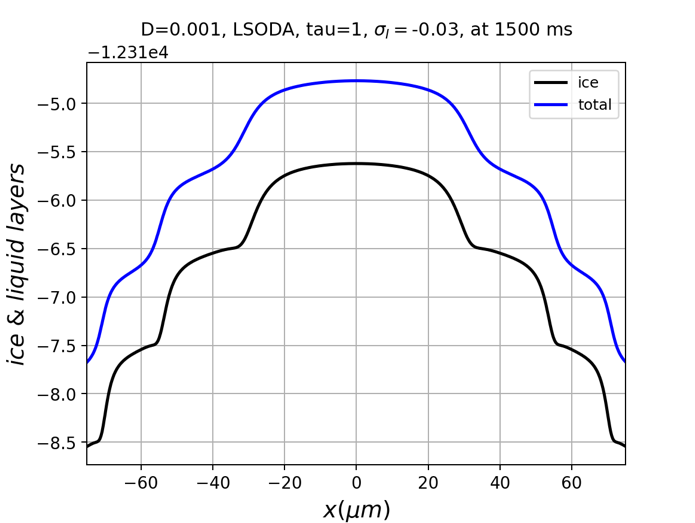
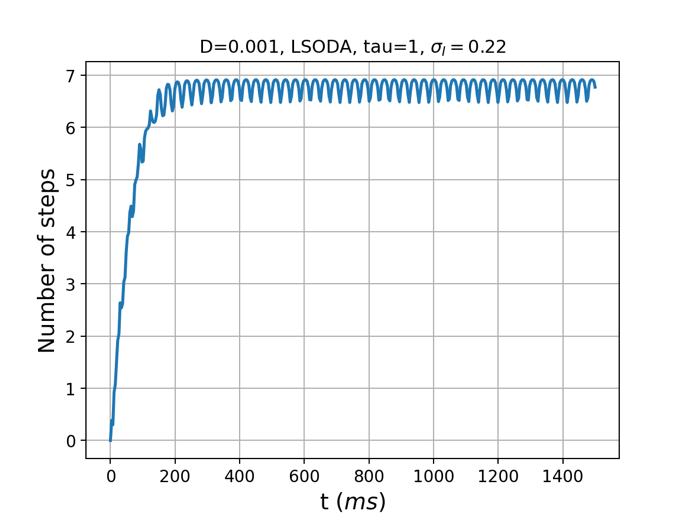
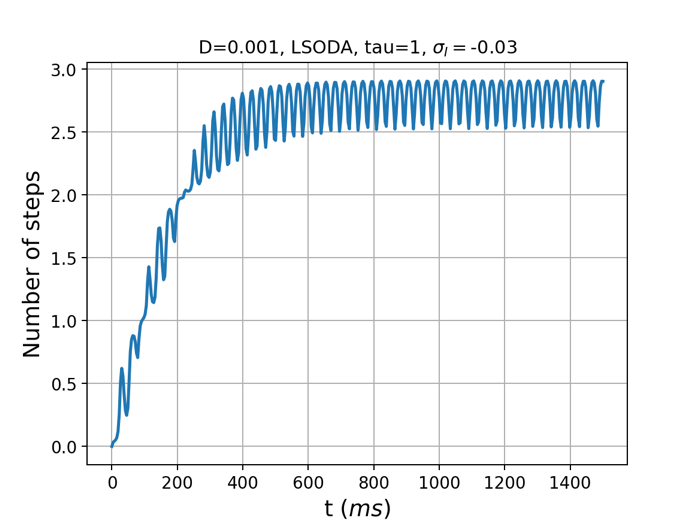
1. **Results**

*I. Existence of faceted ablation*

Figure 3 shows a modeled ice crystal surface under ablating (left) and growing (right) conditions. The ablation scenario was parameterized by subsaturation conditions of the kind shown by the dashed line in Fig. 2, having more extreme subsaturation at corners compared to facet center. We see in Fig. 3 that the facet achieves a steady-state profile – i.e., faceted ablation – after about . The center of the crystal is about three molecular layers thicker than at facet boundaries. The growth scenario, shown on the right of Fig. 3, is a result of parameterizing the water vapor amount in the pattern shown in the solid blue line of Fig. 2. We see that this facet also achieves steady state, although in this case the facet center is about seven molecular layers *thinner* than at facet boundaries.

Both scenarios exhibit signs of diffusive slow-down described in N2016: steps are more tightly bunched together at facet boundaries compared to the facet center. As described in N2016, this bunching is a spontaneous, emergent property of the facet, a result of the fact that, in the case of the growing crystal, supersaturation is most extreme at the corners, hence the rate of new layer formation is greatest there. That bunching at the corners, in turn, reduces the efficiency with which the surface is able to retain deposited water vapor, compared to facet center; eventually, result is that retention is uniform across the facet, and the surface locks into a steady-state profile (i.e. faceted growth).

This ”diffusive slowdown” mechanism was first described in N2016, where only growth conditions were considered. Here, we see that a similar mechanism describes the process of ablation, with layer bunching now reducing the volatility of the surface near the corners.



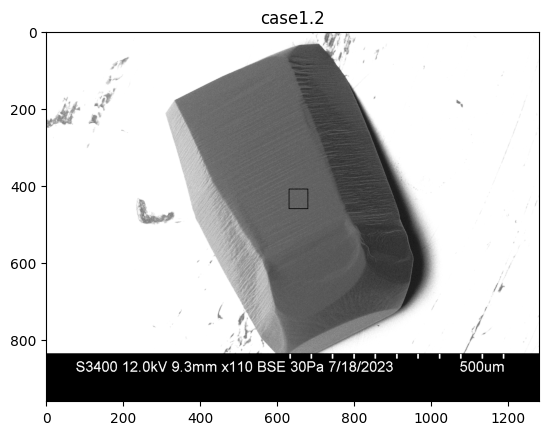
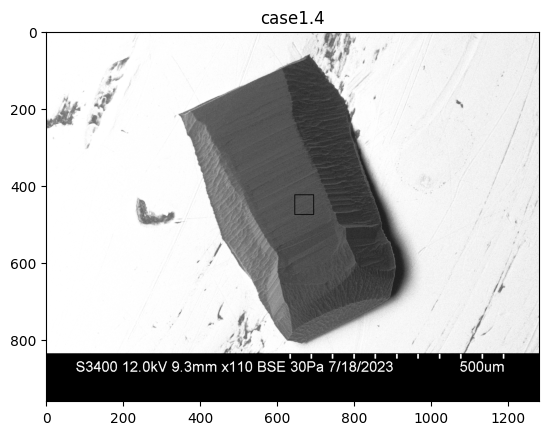
**Figure 3**. Stabilization of faceted ablation (left panels) and faceted growth (right panels).

A useful metric for describing the morphology of steady state profiles such as those appearing in Fig. 3 is the horizontal distance between successive molecular layers. Here we define a mean value of that distance as

(6)

For example, for the ablating facet profile on the left of Fig. 3, whereas for the growing facet profile on the right.

Do real ice crystals exhibit faceted ablation? Figure 4 displays SEM images of an ice crystal under ablating and growing conditions. Since the ablating crystal retains its flat surface, we can conclude that faceted ablation has indeed occurred. The figure shows, moreover, that faceted ablation occurs even when the surface is rough (e.g., the prismatic facets in the figure). In fact, we observe faceted ablation quite frequently in SEM images of ablating ice crystals.



**Figure 4**. An ice crystal under ablating (left) and growing (right) conditions.

*II. Facet concavity and convexity*

The model results shown in Fig. 3 suggest the following general pattern: growing ice facets will exhibit facet concavity, whereas ablating ice facets will exhibit facet convexity. Is this pattern borne out by observations?

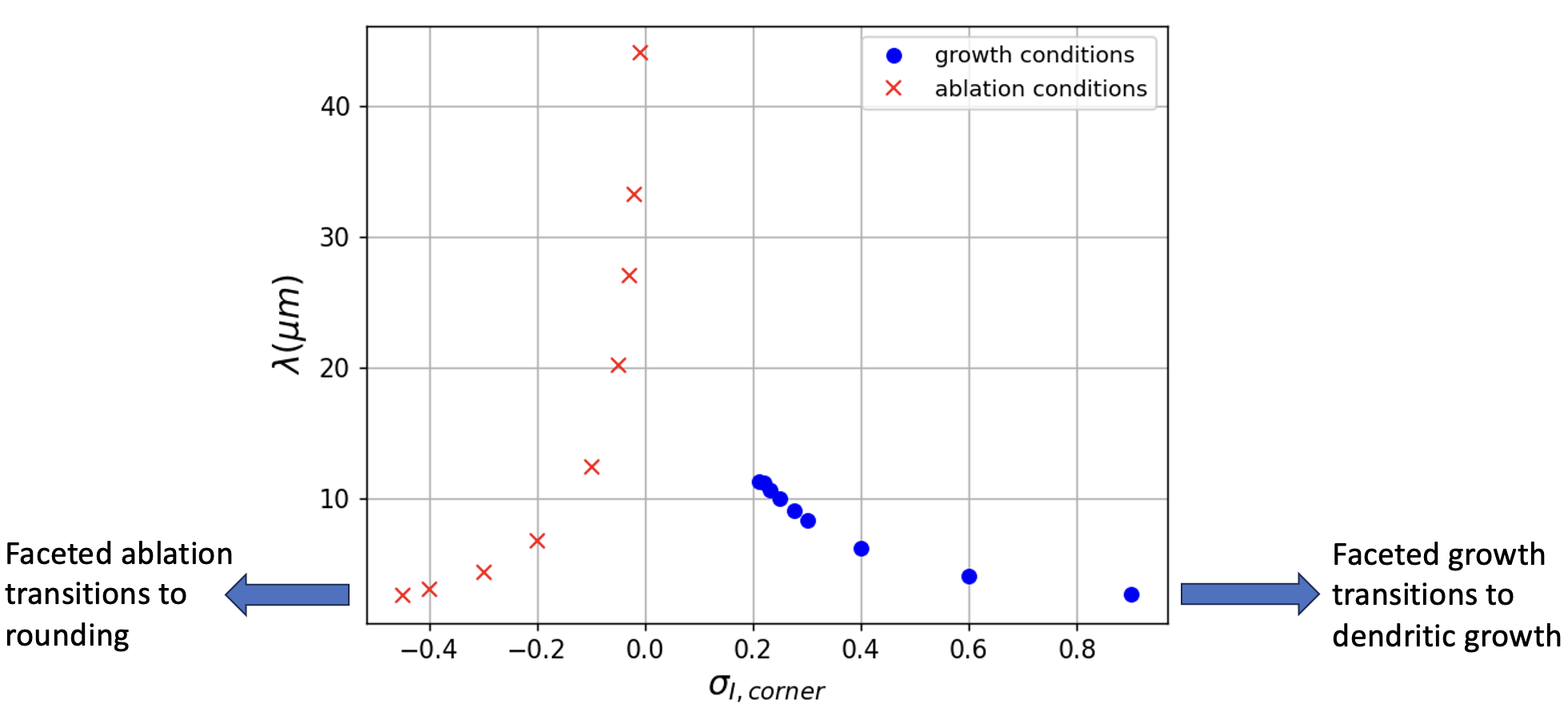
An SEM image of a growing crystal, and a GNBF reconstruction of a portion of its basal facet, are shown in Fig. 5. The GNBF reconstruction reveals a distinct concavity, on the order of 1000s of layers over the horizontal span analyzed (). The crystal is known to be growing, since subsequent images taken of this crystal revealed expanding boundaries against the metal substrate to which the crystal is attached. Thus, the pattern is borne out: growing ice facets do indeed exhibit facet concavity, whereas ablating ice facets will exhibit facet convexity, in agreement with the model prediction.

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| **Figure 5**. SEM image of a growing crystal (left) with its basal surface retrieved (right) | |

*III. Dependence of surface morphology on model conditions.*

Values of the surface layer separation metric, (introduced in Eq. 6), are shown in Fig. 6. These values based on a series of simulations in which the supersaturation at the corner of the crystal () was varied, and all other parameters fixed. Focusing first on the right-hand side, we see that when conditions are just barely supersaturated (relative to surface II, the most volatile microsurface the quasi-liquid is capable of), this distance first appears with . As supersaturation increases to the right, becomes declines monotonically. We surmise that eventually, at high enough supersaturation, growth must cease to be faceted, and become dendritic.

On the left-hand side of Fig. 6 are displayed results when conditions are subsaturated relative to the least volatile microsurface of the model (surface I). We see that under these conditions, steady-state spatial wavelengths first appear much higher – over – and decrease for more extreme subsaturations farther to the left. We surmise that eventually, at extreme enough subsaturation, ablation must cease to be faceted, and become rounded.

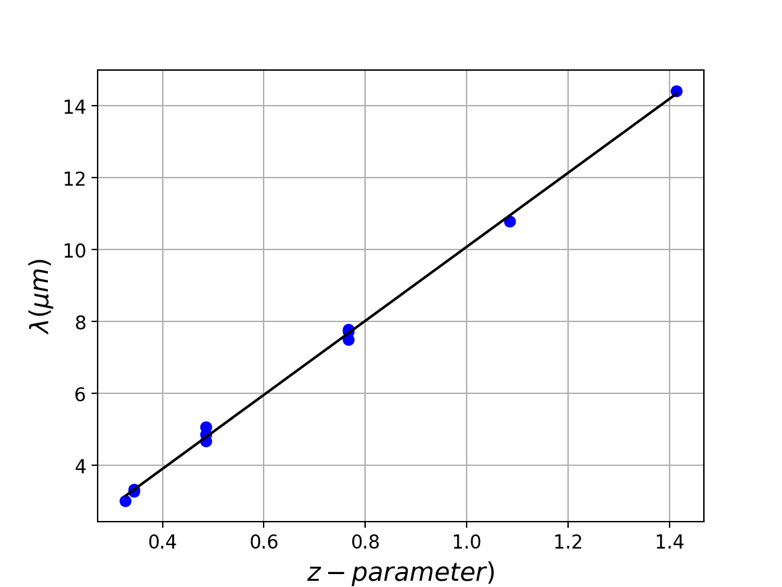


**Figure 6**. Spatial wavelength () as a function of corner supersaturation .

Figure 7 shows the dependence of on a parameter , defined by

(7)

Where (as described above) is the surface diffusion coefficient, is the reduction in supersaturation at facet center relative to facet corners, and is the kinetic deposition velocity. We see that … The observation that is proportional to is noteworthy because of the connection to Turing patterns ….



**Figure 7**. Surface layer wavelength () as a function of parameter for a range of values of , , and . The corner supersaturation is fixed at for all points.

*IV. Facet resilience*

Facet resilience is defined here as the ability of a facet to recover from a perturbation. Actual ice crystal facets exhibit high resilience. For example, in SEM experiments, facets that have been roughened by some perturbation (e.g., by higher temperature) are commonly observed to be restored to smoothness within less than a minute upon reversal of the perturbation. An example is shown in Fig. 8

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| **Figure 8**. SEM image of a rough facet restored to smoothness. |

What resilience, if any, is exhibited by modeled faceted growth and ablation steady states? Figure 9 is an examination of this question. The blue curve is a timeline of , beginning with a flat profile but subject over time to the inverse parabolic supersaturation curve indicated by the solid blue line in Fig. 2. After , it has evolved to the steady-state profile shown, characterized by . After that time, a perturbation is introduced in the form of a new ambient water vapor regime, given by the solid, gray-colored sinusoidal subsaturation curve in Fig. 2. The surface responds by evolving through a transient state that is nearly flat, at , then converting, by at , into a highly rounded profile with a depression at facet center. At that time, the original ambient water vapor regime is once again imposed. By , the surface has recovered its pre-perturbation, steady state.

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| **Figure 9**. Examination of model facet resilience. |

*V. Does facet roughness have an intrinsic characteristic length scale*?

The model presented here is not, unfortunately, capable of the extreme depth variations that mesoscopic roughening represents, because the vertical relief is too great to resolve in our model. There is, however, a hint that the characteristic wavelength of roughness is an intrinsic property of the crystal, rather than the result of some imposed variability (e.g., a repeating variability in the overlying vapor field). That hint is the fact that the model predicts a substantial difference in growth vs ablation values (as shown in Fig. 5), in parallel with the difference between growth vs ablation characteristic distances (as shown in Fig. 4).

*VI. What about differences between facets?* Because facets have distinct underlying crystal cell structures, we can expect that their quasi-liquid properties (in the model, and ) will also be distinctive. Moreover, numerical studies show that the thickness of a single “layer” of ice leads to a proportional increase in .